CUBIC EQUATIONS.

BY PROF. L. G. BARBOUR, RICHMOND, KENTUCKY.

SEVERAL years ago the writer published in the ANALYST (Vol. V, pp 73–79) a method of solving numerical equations of the 3rd deg. with three real roots,—of which two might be equal. The present paper will extend the same general method, so as to solve equations having two imaginary roots. The equations considered are of the form

$$x^3 \pm px \pm q = 0.$$

I. $x^3 - px + q = 0$. The roots are of the form $A + B\sqrt{-1}$, $A - B\sqrt{-1}$, $-2A$. Construct an equation from these roots. We get

$$x^3 - (3A^2 - B^2)x + (A^2 + B^2)2A = 0.$$

Let $B^2 = mA^2$, then

$$x^3 - (3 - m)A^2x + (2 + 2m)A^3 = 0;$$

$$\therefore p = (3 - m)A^2; \quad q = (2 + 2m)A^3;$$

$$\therefore \frac{p^3}{q^2} = \frac{(3 - m)^3}{(2 + 2m)^2};$$

$$\therefore 3 \log p - 2 \log q = 3 \log (3 - m) - 2 \log (2 + 2m).$$

We now make a table with m as the argument, taking it $= .001$, $.002$, &c., successively, up to 3. In the second column write the differences between $3 \log (3 - m)$ and $2 \log (2 + 2m)$. In this table we find the value of $3 \log p - 2 \log q$ in the 2nd column, and thus get the value of m in the 1st column, as we find a number from its logarithm in the ordinary logarithmic tables. Then since $(3 - m)A^2 = p$, $A = \sqrt{[p \div (3 - m)]}$; $B^2 = mA^2$; $B = A\sqrt{m}$. Test: $(2 + 2m)A^3 = q$, $\therefore \log A = \frac{1}{3} \log q - \frac{1}{3} \log (2 - 2m)$.

II. $x^3 - px - q = 0$. The roots are $-A - B\sqrt{-1}$, $-A + B\sqrt{-1}$, $+2A$. The equation constructed from these roots is

$$x^3 - (3 - m)A^2x - (2 + 2m)A^3 = 0;$$

$\therefore p = (3 - m)A^2$ and $q = (2 + 2m)A^3$, as before. The difference is that when, as in this case, q is negative, the signs of the roots will be reversed, as above indicated.

III. $x^3 + px + q = 0$. Construct the equation from the roots $A + B\sqrt{-1}$, $A - B\sqrt{-1}$, and $-2A$. We again get

$$x^3 - (3A^2 - B^2)x + (A^2 + B^2)2A$$
$$= x^3 - (3 - m)A^2x + (2 + 2m)A^3.$$

But since p in this case is positive, it is necessary and sufficient to write

$$x^3 + (m - 3)A^2x + (2 + 2m)A^3 = 0.$$

In cases I and II, $m < 3$, or at most, $m = 3$. In case III, $m > 3$ or at least, $m = 3$. In fact there will be one place common to the two tables, viz., when $m = 3$. Then $3 \log(3—m)$, or, in the other table, $3 \log(m—3) = 3 \log 0 = \infty$; and the tabular method cannot be used. The solution in this instance, however, is easy enough, for $p=(3—m)A^2 = 0$, and the equation becomes $x^3 + q = 0$; from which the roots are readily found.

This second table is constructed from the formula $3 \log p - 2 \log q = 3 \log(m—3) - 2 \log(2+2m)$. Ascending values are assigned to m from 3 up as high as may be needed.

IV. $x^3 + 70x - q = 0$. The roots are $-A+B\sqrt{-1}, -A-B\sqrt{-1}$, and $+2A$; $\therefore\ x^3+(m-3)A^2-(2+2m)A^3 = 0$. The same table is used as in case III.

General Remarks.

If p is $+$, in the equation under consideration, there must be 2 imaginary roots. But if p is —, there may be 3 real, unequal roots, or 3 real roots, of which two are equal; or one real root and two imaginary and unequal; or one real root and, perhaps we may say, 2 equal imaginary roots. The discrimination is effected thus: If $(p^3 \div q^2) > \frac{27}{4}$, i. e., $3 \log p - 2 \log q > .8293038$, there are 3 real, unequal roots. If it be $< .8293038$ there are 2 imaginary unequal roots. But if it $= .8293038$, two of the roots are equal and are ordinarily considered real. However, if we choose to regard B as $= 0$, we may treat this as coming under the head of imaginary roots; $A+B\sqrt{-1}$, and $A-B\sqrt{-1}$ becoming equal when $B = 0$.

Proof:—It is evident that $3 \log(3—m) - 2 \log(2+2m)$ reaches a maximum when $m = 0$. Then $3 \log 3 - 2 \log 2 = \log \frac{27}{4}$.

In my former article, the table was constructed by the formula $3 \log(1+a+a^2) - 2 \log(a+a^2)$; $\therefore$ by differentiating for maximum and minimum, $a = 1, -\frac{1}{2}$ or -2. It was shown that when $a = 1$, there are two equal and real roots; and the same is true when $a = 2$ or $-\frac{1}{2}$. In our present inquiry, when $m = 0$, $mA^2 = B^2 = 0$, $\therefore$ the roots are $A, A, -2A$; or $-A, -A, +2A$.

The method of procedure then is this: Observe the sign of p in the given equation; if it is —, then find $3 \log p—2 \log q$, either in the old table for 3 real roots, which extends from $+\infty$ down to $\log \frac{27}{4} = 82930772831$; or else in the first of the new tables subjoined, which extends from .82930772-831 down to $-\infty$. If it falls in the old table, the argument is a; the roots were called r, s, and t; $r^2 = p \div (1+a+a^2)$, $\therefore \log r = \frac{1}{2} \log p - \frac{1}{2} \log(1+a+a^2)$; $s = ar$; $t = r+s$, but with the opposite sign.

The signs of r, s, and t are determined by that of q. If q is $+$, r and s

are +, and t — If q is —, r and s are —, and t +. In other words r and s always have the same sign as q; t has always the opposite sign. The negative values of a are not employed, since they give no new results.

In the same way if q is +, the imaginary roots are $+A+B\sqrt{-1}$, and $+A-B\sqrt{-1}$. The real root, corresponding to t, is $-2A$. But if q is —, the roots are $-A-B\sqrt{-1}$, $-A+B\sqrt{-1}$, and $+2A$.

Again if p is +, look only in the second subjoined table, No. III, extending from $-\infty$ up as far as we need it.

It is interesting to note that if p is —, we may by changing q arbitrarily, in value and sign, get an equation with three real roots, all unequal, or two equal; or one real and two imaginary; but if p is +, there is no help for it; two roots must be imaginary.

In table II, m is the argument; $(3-m)A^2 = p$; $mA^2 = B^2$;

$$\therefore \log A = \tfrac{1}{2}[\log p - \log(3-m)],$$
$$\log B = \log A + \tfrac{1}{2}\log m.$$

In table III, m is the argument; $(m-3)A^2 = p$;

$$\therefore \log A = \tfrac{1}{2}[\log p - \log(m-3)],$$
$$\log B = \log A + \tfrac{1}{2}\log m.$$

The use of the tables is best explained by actual examples. In the former article one or two examples were worked out in full. It was found that the results were true for 5, 6, or 7 places in equations having three real roots. I have very recently tried the method on a few problems from Prof. Wentworth's "Complete Algebra." On p. 508, under the head of Trigonometric Solution of Equations, he says "Take the difficult equation

$$x^3 - \tfrac{403}{441}x + \tfrac{46}{147} = 0."$$

Solving by trigonometry he gets $x' = 0.42855$, $x'' = 0.66670$, $x''' = -1.09525$.

By the method proposed in this paper, $3\log\frac{403}{441} - 2\log\frac{46}{147} = .8917182$. From the table we get, by proportional parts, $a = .6428582$, $\therefore r = .66666626$; $s = .42857182$; $t = -1.09523808$. The exact roots are

$$r = \tfrac{2}{3} = .666\,;\ s = \tfrac{3}{7} = .428571428571\,;\ t = \tfrac{23}{21} = 1.09523809.$$

The value of t is usually the most exact. By a simple interpolation in the table, the value of a can be found more closely, giving the roots a little nearer. Since $s = ar$, $a = s \div r$, we find the exact value of $a = \frac{3}{7} \div \frac{2}{3} = .6428571428571+$.

On p. 510 Prof. W. gives as "the true values," 0.42857, 0.66667, —1.09524.

One defect of the method of this article is that it gives results true to only 5 or 6 or 7 places. By an extension of the method closer results could be

obtained, but they are seldom needed. It is seen above that the actual results are nearer true than those which Prof. W. deems sufficiently exact.

On p. 496 he says "In rare cases two of the roots are so nearly equal that Horner's Method carried out as above will not find them both. Take for example $x^3+11x^2-102x+181=0$,
in which we have found that there are two roots between 3 and 4 Horner's method gives, for the first transformed equation,

$$y^3+20y^2-9y+1=0.$$

Since the coefficient of y^2 is large, it is best to neglect y^3 only." That is, the equation $20y^2-9y+1=0$ must be solved,

This plan, if my memory serves me, is employed by Newton in similar cases. From curiosity I tried this problem by the aid of the tables. At the outset we encounter the unavoidable difficulty of removing the second term of the equation. This defect is inseparable from the method. The removal can be accomplished, however, without much trouble, by Horner's Synthetic Divisor, or by substitution. Then by the regular use of the table the roots are found to about 5 or 6 places. One root is 3.22953. Prof. W. gives 3.22952. A curious application of the tabular method has been made to a problem in Newcomb's Algebra, p. 192.

$$\frac{\sqrt{(x+a)}}{\sqrt{(x-a)}}=\frac{x}{a}$$

This gives $x^3-ax^2-a^2x-a^3=0$,
or removing the second term,

$$y^3-\tfrac{4}{3}a^2y-\tfrac{38}{27}a^3=0,$$

in which case $x=y+\frac{1}{3}a$. Also $p=-\frac{4}{3}a^2$; $q=-\frac{38}{27}a$. Therefore

$$p^3\div q^2=\tfrac{64}{27}\div(\tfrac{38}{27})^2,$$

in which a is eliminated. Proceeding then as usual, we get $3\log\frac{4}{3}-2\log\frac{38}{27}=.0779765$. Since p is —, we look in the 2nd table, and find m bet'n .648 and .649. By proportion, $m=.64833392$; $3-m=2.35166608$. Log $p=.1249387$. Log $(3-m)=.37137566$. Difference $=\bar{1}.75356304$. Log $A'=\frac{1}{2}[\log p-\log(3-m)]=\bar{1}.87678152+\log a$; $A'=.7529767a$ Log $B'=\log A'+\frac{1}{2}\log m$. $B'=.6062906a$; $2A'=1.5059534a$. Hence the roots of the equation

$$y^3-\tfrac{4}{3}a^2y-\tfrac{38}{27}a^3=0$$

are $-.7529767a-.6062906a\sqrt{-1}$, $-.7529767a+.6062906a\sqrt{-1}$, and $+1.5059534a$. The roots of the original equat's are found by adding $\frac{1}{3}a=.333a$ to each of these. We get $-.4196434a-.6062906a\sqrt{-1}$, $-.4196434a+.6062906a\sqrt{-1}$, and $+1.83928671a$.

Let us now test one of these roots, say the last given. By substitution.

$$\sqrt{(1.83928671a+a)} \div \sqrt{(.83928671a)} = 1.83928671a \div a,$$

or $$\sqrt{(2.83928671)} \div \sqrt{(.83928671)} = 1.83928671.$$

$\therefore\ \frac{1}{2}\log 2.83928671 — \frac{1}{2}\log .83928671$ should $=\log 1.83928671.$

Now $\frac{1}{2}\log 2.83928671 = .22660925$

and $\frac{1}{2}\log\ .83928671 = \bar{1}.96195515$

Diff. $= .2646541$

and $\log 1.83928671 = .2646495$

Log. error $.0000046$

In numbers, 1.8393063 — 1.8392867 = 0.0000196.

Table II.

$x^3 — px \pm q = 0.$

m	$(3-m)A^2=p$
.0	.8293038
.1	.7023486
.2	.5810516
.3	.4641448
.4	.3506039
.5	.2395774
.6	.1303336
.7	.0232256
.8	—1.9146631
.9	—1.8070907
1.0	—1.6989700
1.1	—1.5897622
1.2	—1.4789121
1.3	—1.3658311
1.4	—1.2498776
1.5	—1.1303339
1.6	—1 0063774
1.7	—2.8770436
1.8	—2.7411677
1.9	—2.5973221
2.0	—2.4436975
2.1	—2.2779441
2.2	—2.0969100
2.3	—3.8962063
2.4	—3.6694360
2 5	—3.4067139
2.6	—3.0915150
2.7	—4.6929004
2.8	—4.1414628
2.9	—5.2158108
3.0	— ∞.

Table III.

$x^3 — px \pm q = 0.$

m	$(m-3)A^2=p$
3.0	— ∞.
3.1	—5.1723722
3.2	—4.0545314
3.3	—4.5623668
3.4	—4.9172146
3.5	—3.1884250
3.6	—3.4068782
3.7	—3.5890383
3.8	—3.7447276
3.9	—3.8802753
4.0	—2.
4.1	—2 1069777
4.2	—2.2034770
4.3	—2.2912184
4.4	—2.3715364
4.5	—2.4454885
5.	—2.7447275
6.	—1.1391077
7.	—1.3979400
8.	—1.5863650
9.	—1.7323938
10.	—1.8504487
11.	—1.9488476
12.	0.0327809
13.	0.0556855
20.	1.4448481

Use of Tables.—In Table II, find m; then $(3-m) \times A^2 = p$; $\therefore \log A = \frac{1}{2}[\log p — \log(3—m)]$.
$B^2 = mA^2$, $\therefore \log B = \frac{1}{2}\log m + \log A$.

In Table III, find m; then $(m—3)A^2 = p$; $\therefore \log A = \frac{1}{2}[\log p — \log(m—3)]$. B^2 = as in T. II.

The accuracy of the work may be tested by the formula, $\log A' = \frac{1}{3}[\log q - \log(2+2m)] = \bar{1}.87678151$, the value obtained above being $\bar{1}.87678152$

The equation $y^3 - \frac{4}{3}a^2y + \frac{15}{27}a^3 = 0$ has 3 real roots and may be solved by the old table. In general, if the exponent of the literal part of the absolute term be $\frac{3}{2}$ the exponent of the like literal part of the term containing the 1st power of the unknown quantity, the present method will solve the equation. Thus such equations as $x^3 \pm \frac{3}{4}a^{2/3}x \pm \frac{5}{6}a = 0$, or as $x^3 \pm \frac{3}{5}a^{3\div 7}x \pm \frac{8}{11}a^{9\div 14} = 0$, are easily managed. Also such as

$$x^3 \pm \frac{\sqrt{11}.\sqrt[3]{13}}{\sqrt[5]{15}.\sqrt[7]{19}}x \pm \frac{\sqrt[11]{23}.\sqrt[17]{27}}{\sqrt[13]{25}.\sqrt[19]{29}} = 0$$

offer no difficulty.

Considering the table previously furnished (see p. 79, Vol. V) as No. I, we designate the foregoing tables as Nos. II, and III. No. II is calculated by the formula $3\log(3-m) - 2\log(2+2m)$; No. III, by the formnla $3\log(m-3) - 2\log(2+2m)$. In No II, p is $-$; in No. III, p is $+$; and in both cases there are two imaginary roots.

ANOTHER SOLUTION OF PROB. 435 (SEE P. 94), BY R. J. ADCOCK.—Let m^2 = the number of points, arranged uniformly in any manner, on a unit of surface, then m = average number on a unit of length. The number of bases MN (see figure on p. 94) on chord AB equals the number of positions of two points on AB, $= \frac{1}{2}m^2AB^2 = \frac{1}{2}m^2c^2 = 2m^2r^2\sin^2\theta$. (Math. Monthly, No. 1, Vol. I.) Hence the total number of triangles with bases on AB is $2m^2r^2\sin^2\theta \times \pi m^2r^2 = 2\pi m^4r^4\sin^2\theta$.

While AB passes through the number of positions $mrd.\cos\theta$, the number of triangles on it will be $2\pi m^5r^5\sin^3\theta d\theta$. And it will be in all its positions while θ changes from 0 to π. Hence.

$$\int_0^\pi 2\pi m^5r^5\sin^3\theta d\theta = 2m^5r^5(-\tfrac{1}{3}\sin^2\theta\cos\theta - \tfrac{2}{3}\cos\theta) + C = \tfrac{8}{3}\pi m^5r^5$$
$$= \text{number of triangles.}$$

The number of constant bases $MN = y$, on AB is $m(c-y)$, and their sum is $m(c-y)y$. Hence the sum of all bases on AB is

$$\int_0^c m^2(c-y)ydy = \tfrac{1}{6}m^2c^3 = \tfrac{4}{3}m^2r^3\sin^3\theta.$$

Now $\frac{1}{2}m$ multiplied by the square of perpendicular to AB from any point in the circumference equals sum of altitudes in that perpendicular. Therefore the sum, of all altitudes of each triangle having base on AB, $\frac{1}{2}(m^2\div\pi)$ multiplied by the sum of the two volumes described about AB as an axis by the two segments of the circle made by chord AB; that is

$m^2r^3(\pi\cos\theta+2\sin\theta-2\theta\cos\theta-\frac{2}{3}\sin^3\theta)=$ sum of altitudes for each base on AB. Hence $\frac{2}{3}m^4r^6(\pi\cos\theta+2\sin\theta-2\theta\cos\theta-\frac{2}{3}\sin^3\theta)\sin^3\theta=$ sum of triangles on AB. And

$$\int_0^{\frac{1}{2}\pi}\tfrac{4}{3}m^5r^7(\pi\sin^4\theta\cos\theta+2\sin^5\theta-2\theta\sin^4\theta\cos\theta-\tfrac{2}{3}\sin^7\theta)d\theta$$

$$=\tfrac{2}{3}m^5r^7\left\{\tfrac{1}{5}\pi\sin^5\theta-\frac{12}{5}\left(\tfrac{1}{5}\sin^4\theta+\frac{1.4}{3.5}\sin^2\theta+\frac{1.2.4}{1.3.5}\right)\cos\theta-\tfrac{2}{5}\theta\sin^5\theta\right.$$

$$\left.+\frac{2}{3}\left(\tfrac{1}{7}\sin^6\theta+\frac{1.6}{5.7}\sin^4\theta+\frac{1.4.6}{3.5.7}\sin^2\theta+\frac{1.2.4.6}{1.3.5.7}\right)\cos\theta\right\}+C$$

$$=\tfrac{2}{3}m^5r^7.\frac{1024}{525}.\quad\text{Dividing by }\tfrac{8}{3}\pi m^5r^5\text{ gives }\frac{256r^2}{525\pi}$$

for the average area, the same as found by Mr. Seitz in his corrected result. [See *Errata*, p. 128.]

RECONSIDERATION OF SOLUTION OF PROB. 239. (*P.* 48, *VI.*)

BY CHAS. H. KUMMELL, U. S. COAST SURVEY, WASH., D. C.

THERE were three solutions of this problem published, the first one furnished by me and supported by the Editor,* the second by Mr. Adcock and the third by Prof. P. E. Chase. A thorough investigation of the law of error in two dimensions, which I am making at present, has however convinced me that Prof. Chase's solution is the only correct one. In this solution, formulæ are used which Sir John Herschel developed in his Lecture on target shooting. The proof he gives, though I now admit it to be perfectly correct, failed to convince me of my error. Recently I thought of investigating the matter from a new point of view. I regarded shooting as compounded of two independent operations, viz., sighting and leveling, and I assume that errors in sighting, i. e., deviations x from the vertical y-axis and errors in leveling, i. e., deviations y from the x-axis each follow the ordinary law of error, so that if $\varepsilon_x=$ mean error of sighting and $\varepsilon_y=$ mean error of leveling, then

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*It will be seen, by referring to p. 50, Vol. VI, that our "support" of Mr. Kummell's solution was *conditional*. We there stated, and still assert, that the solution is correct "if the equation $y=ce^{-h^2x^2}$ represents the relation between an error and its probability."

It is obvious, however, that, in target shooting, the equation does not represent that relation; for it is apparent, from a mere statement of the case, that the center of the target is not as likely to be hit, by any single shot, as a contiguous concentric circle; and hence a very small deviation from the center, of a single shot, is not the most probable.—Editor.